How solitary waves collide in discrete granular alignments

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Solitary waves in continuum media pass through each other with only a slight phase change. However, in an intrinsically nonlinear many-body system such solitary waves could behave differently. It was predicted and experimentally confirmed that head-on solitary wave collisions in granular alignments are followed by the formation of tiny secondary solitary waves in the vicinity of the collision point. While it remains a challenge to provide an analytical treatment of the local time evolution, we present arguments and associated simulations to address a crucial unknown, namely, why the secondary solitary waves must form. Extensive numerical investigations on solitary wave collisions at a grain center and at an edge show marked differences. The effects of softening the grain repulsion are discussed to validate the arguments.

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I. INTRODUCTION

The study of nonlinear waves in continuum systems such as in liquids has a long history [1]. Solitary waves (SWs), which are compact propagating bundles of energy, are a kind of nonlinear wave that has been the subject of many studies [1–3]. Existing theory contends that when two identical and opposite propagating SWs meet, they pass through each other suffering only a phase shift [4]. However, some old experimental evidence already exists that claims that weak *dispersive* waves may be generated from the collision region of SWs [5].

Propagation of SWs in discrete alignments of grains has attracted significant recent attention [6–8]. Collision of identical and opposite propagating SWs in granular alignments show that postcollision SWs are different from precollision SWs. The former carry more energy than the latter [9,10]. Dynamical simulations using energy conservation reveal that some energy is retained in the collision region shortly after a collision. This energy leads to the formation of weak or secondary solitary waves (SSWs) in the vicinity of the collision point. Now experimentally confirmed [9], these *nondispersive* SSWs were seen earlier in simulations [10] and is a well-established phenomenon in nonlinear dynamics of discrete systems. This paper focuses on why these SSWs must form and its consequences. It is perhaps relevant to note that understanding the process of SSW creation and control is a difficult problem and the objective of the paper is to develop a qualitative picture to explain numerical and experimental observations.

At this time there is no exact solution that exists to even describe a propagating SW far from the edges in a granular alignment. Continuum approximation [6] and an approximate solution for grain displacement, velocity, and acceleration are all that are available [8]. Both solutions are valid far from the boundaries where dynamics is quite different [11]. Hence, arguments and simulations are our only options for probing SW collisions and SSW formation at this stage. However, successful construction of the correct qualitative picture to describe SW collisions could set the stage for a future theory even in the absence of an exact solution.

This paper is organized as follows. The physics model is described first. We next argue that the outcome of SW collisions must depend on the characteristics of the boundary and the softness of the potential. These expectations are then shown to be confirmed through dynamical simulations. We conclude by suggesting that SSW formation signifies the possible existence of a generalized equilibrium state in confined nonlinear chains.

II. MODEL AND METHOD

Let R and m be the radius and mass, respectively, of each spherical elastic grain placed in locations $z_1, z_2, \ldots$ in an alignment in such a way that the dynamics is translational. We assume that Young's modulus Y and Poisson's ratio σ describe the elastic grains. The grains barely touch one another at the initial time $t=0$ and the system is assumed to be held within fixed end walls. If any two spheres i and $i+1$ come in intimate contact, they repel according to the Hertz law [12], $V(\delta_{i,i+1})=a\delta_{i,i+1}^{5/2}$ where $\delta_{i,i+1}\equiv 2R-(z_{i+1}-z_i)$, and $a=\frac{2}{5D}\sqrt{\frac{R}{2}}$, and $D=\frac{3(1-\sigma^2)}{2Y}$. In general, instead of considering spheres with $V(\delta_{i,i+1})=a\delta_{i,i+1}^{5/2}$, one can insert n (≥ 2) instead of 5/2 for the exponent of $\delta_{i,i+1}$ along with an appropriate change in the dimensions of a such that V has the units of energy. For $n\to 2$ one finds a "soft" nonlinear potential with $n=2$ being where nonlinear effects vanish. When $n\to\infty$, the potential behaves like a one-sided hard-core potential.

The equation of motion for each grain (except the boundary grains) is

$$m\frac{d^2 z_i}{dt^2}=na(\delta_{i,i-1}^{n-1}-\delta_{i+1,i}^{n-1}),\quad n\geq 2. \qquad (1)$$

We carry out dynamical simulations via the third-order Gear algorithm [13]. We choose 10^{-5} m, 2.36×10^{-5} kg, and 1.0102×10^{-3} s as the units of distance, mass, and time, respectively. The integration time step dt used was small enough to resolve the finer details while allowing runs over many decades in time. Typically for $n=2.5$ we use $dt=5\times10^{-7}$ and for $n=2.1$, $dt=10^{-6}$. The grain diameter is set to 100, i.e., 1 mm, $m=1$ and $a=5657$ [8], a value ($=4.14\times10^{7}$ $\mathrm{N/m^{3/2}}$) which is in the range of elasticity of silicate materials. In order to compare to previous work (see Ref. [10]), we also perform simulations using $a=1$ when we consider $n<5/2$. In the calculations presented below, we use $N=500$ for even chains and $N=499$ for odd chains. In our simulations, energy is conserved to an accuracy of 0.004% over 10^6 time steps. SSWs are generated in the odd and even chains using different initial velocities.

III. THEORETICAL CONSIDERATIONS

Observe that there is no characteristic length scale for Taylor expansion of the right-hand side (RHS) of Eq. (1) in terms of $\delta_{i,i\pm1}$. Therefore, it is not possible to obtain a quadratic dependence of $\delta_{i,i\pm1}$ on the RHS. Consequently, no acoustic propagation is admissible in this system [6,8,14]. So, no perturbation can spread from one grain to the next via *sustained oscillations* of the grains. Hence, the perturbation cannot spread into the system as it does in a chain of harmonic oscillators, or as in a chain with both harmonic and anharmonic nearest-neighbor interactions [15].

Due to the nonlinear nature of the grain-grain repulsion, we suggest that energy transport between grains can only happen during the short contact times between them. Virial theorem [16] applied to our system yields the following result: $2\langle E_{\mathrm{kin}}\rangle=n\langle E_{\mathrm{pot}}\rangle$ (where $\langle\cdots\rangle$ is the time-averaged total kinetic energy) or that $\langle E_{\mathrm{kin}}\rangle=\frac{n}{n+2}E$, where E is the total energy imparted via some initial perturbation. For spherical grains $\langle E_{\mathrm{kin}}\rangle=\frac{5}{9}E$. This result is confirmed in our simulations and detailed elsewhere [11].

Since no acoustic propagation is allowed, and energy transport must transpire, an allowed mechanism for satisfying both the conditions might be the presence of a propagating energy pulse. Simulations confirm that any perturbation initiated in the system described by Eq. (1) must propagate as a SW [11].

Since Hertz law describes the repulsive force arising from small deformations of elastic grains, the dynamics relates to pulse propagation at speeds that are orders of magnitude smaller than sound speed in the elastic grains. As in nonlinear waves, the pulse speed depends upon the perturbation details and the pulse width varies with n, being 1 grain diameter across if $n\to\infty$ and with the width $\to\infty$ if $n\to2$ [14]. For practical purposes, the quantity $\langle E_{\mathrm{kin}}\rangle$ serves as a way of determining whether a solitary wave has formed.

We now qualitatively explain why SSWs must form. Let us first argue why SWs must break and reform during a collision event. If L is the width and v is the speed of the SW, then $\tau=\frac{L}{v}$ defines the time taken for the transfer of mechanical energy between the leading and the trailing edges of the SW. Thus, when two identical and opposite propagating SWs with each involving several grains collide across times $\Delta t<\tau$, the grains in the two leading edges of the colliding SWs would be traveling in opposite directions to the grains trailing these. Causality requires finite time must be taken by SWs to pass through each other or, equivalently, for a SW to reverse direction. Since across times $\Delta t<\tau$ the original SWs cannot be stable, after the collision the SWs must be *replaced* by newly formed SWs. So, the precollision and postcollision SWs may not be identical in their properties.

TABLE I. Differences between leading SSWs in odd and even chains (see text). $E_{\mathrm{kin}}\%\equiv(E_{\mathrm{kin\ max\ SSW}}/E_{\mathrm{kin\ max\ SW}})\times100$ is measured at grain 150.

Chain	c_{SW}	t_c	c_{SSW}	$v_{\mathrm{max\ SSW}}/10^{-6}$	$E_{\mathrm{kin}}\%$
Odd	8.4	24.684	3.80	6.397	0.035
Even	8.4	24.746	4.68	18.304	0.288

We now turn our attention to why SSWs form. Virial theorem dictates the kinetic and potential energies that can be carried by a SW. In earlier work [11] we have argued that the time-averaged width of a SW is 7 grain diameters, where the velocities of the leading or trailing grains and the central grain vary by 6 orders of magnitude (Table I of [11]). The finiteness of the width of the SW is often referred to in the literature as the existence of *compact support*. For the sake of simplicity let us consider the ratios of the grain velocities in a SW when it is away from the boundaries and ignore the grains moving at velocities that are an order of magnitude or more smaller than the velocity of the central grain. Table I of Ref. [11] reveals that the velocity ratios would be roughly 0.3:1:0.3 with the SW being about 3 grain diameters wide in this crude approximation. When this 3-grain packet collides with a boundary and the leading grain starts moving in opposition to the trailing ones, the former SW can be thought of as being *squeezed*. Such squeezing would naturally raise the potential-energy content of the bundle beyond what is dictated by the virial theorem and cannot be stable. To lower this excess potential energy, the only option the bundle has is to involve interactions with more grains as it turns around. Simply put, we have just argued that the squeezing of the SW must cause it to dilate. Since compact support must be realized once the bundle is away from the boundary, and there is no mechanism that allows taking back of the momentum transferred to the grains, the system has no choice but to form a returning SW with less energy than its precollision predecessor and one or more smaller SWs, which we call SSWs. As we shall see, simulations presented later indeed confirm our argument. It is not surprising hence that SSW production crucially depends on the nature of the boundary. In the case of collisions where a central grain remains static at all times, some of the excess potential energy caused by the squeezing of the SW is transferred to this central grain. Hence, SSW production is significantly reduced in collisions involving a static central grain. This is not the case when the collision happens at a grain edge where no potential energy can be temporarily stored and thus, as one would expect, more SSWs are produced at such collisions.

IV. DYNAMICAL SIMULATIONS

We let two identical impulses directed into the chain be initiated at $t=0$ by assigning $v_1(0)=v_0$ and $v_N(0)=-v_0$ at the two ends of the granular alignments. These impulses lead to the formation of identical SWs [11]. We first present our studies for even-N (even-chain) and odd-N (odd-chain) systems. The SWs approach and cross at the edge of the two central grains in the even chain and at the center of the central grain in the odd chain. After the crossing occurs, the SWs continue toward the opposite ends of the chain. Our focus is on the detailed dynamics of the crossing event. In the absence of a two-SW solution, this dynamics cannot be probed analytically. Experimental investigation of the process is also challenging because (see, for example, [9]) the effects are subtle and dissipative forces complicate measurements. This is the reason why we ignore dissipation in our calculations. Effects of dissipation [17] can always be incorporated when making direct comparison with specific experiments (again see [9]).

V. ODD AND EVEN CHAINS

As argued above, we find that SSWs are generated soon after the two SWs meet and travel through one another. The SSWs generated in odd and even chains turn out to be different. Extensive simulations have been carried out assuming $n=2.5$ and $a=5657$ and $v_0=\pm 5\times 10^{-4}$. We track the time evolution of the grain displacements and velocities and compute the velocities of propagation of the SWs (c_{SW}) and the SSWs (c_{SSW}). In addition, we compute the total kinetic and potential energies as functions of time. Figure 1 shows the space and time dependences of the total kinetic energies of the traveling SWs in an even and in an odd chain, respectively. SSW formation commences immediately after collision of the SWs and the energy of the first SSW is noticeably smaller in the odd chain when compared to the same in the even chain. Table I summarizes the values of c_{SW}, the instant of the collision t_c, c_{SSW}, and the velocity of the fastest-moving grain within the SSW, $v_{\max\ \mathrm{SSW}}$, in each case and shows that the SSW in the even chain carries 8.2 times more kinetic energy than in the odd chain.

In addition to the above study, we have considered studies with $v_0=10^{-4}$, 5×10^{-5}, 3×10^{-5}, and 10^{-5}. Our calculations show that the kinetic energies of the *fastest-moving* SSW in the even chain are 8.20, 7.86, 8.44, and 9.08 times more than the kinetic energies of the fastest-moving SSW in the odd chain. Given that the dynamics in these systems is highly sensitive to the details of the initial conditions, and that the SW itself is a moving object with some inevitable fluctuations in its length, the kinetic energy carried by the fastest-moving grain of the leading SSW in the even chain is consistently ≈ 8 times larger. Recall that SSWs produced due to head-on collision carry very small amounts of energy and may be difficult to experimentally detect. However, if direct detection is to be attempted, SSW formation in edge collisions may be easier to detect by nearly an order of magnitude.

Our studies also show that in spite of the differences in the kinetic energies of the largest SSWs between the even and odd cases, the actual amounts of energy used by the odd and the even chains to make all the SSWs produced in each chain are not very different. We find that $\approx 2.9\%$ of the total

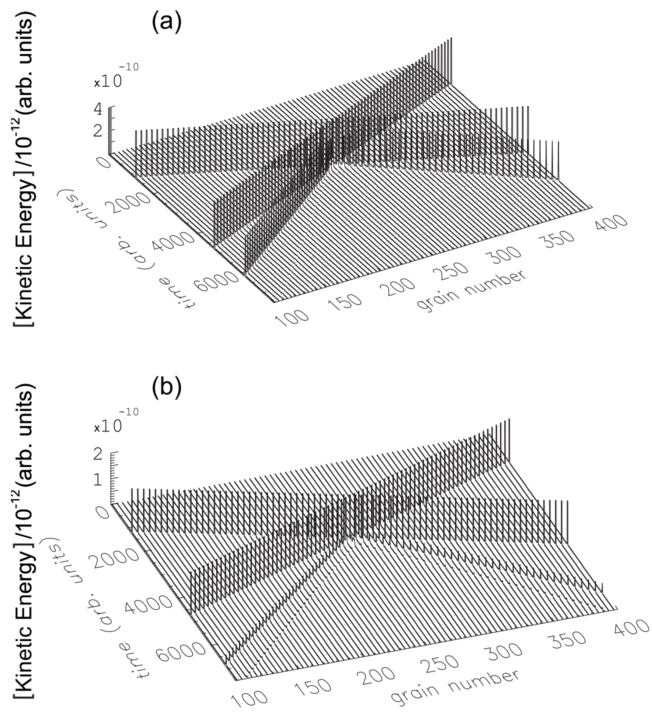

FIG. 1. SSWs are created after the crossing of two identical SWs traveling in opposite directions in (a) an even chain ($N=500$) and (b) an odd chain ($N=499$). Only parts of the original SWs are visible. Observe that the energy content of the SSW in the even case is significantly larger than that in the odd case. Kinetic energy of the grains (in arbitrary units), time (in arbitrary units), and grain positions are shown along the z, y, and x axes, respectively.

kinetic energy of the original SWs was used to make the SSWs for the even case and $\approx 2.3\%$ of the total kinetic energy was used to make the SSWs for the odd case. Observe that virial theorem allows us to readily obtain the total kinetic energy from the total energy and hence $\approx 2.9\%$ and $\approx 2.3\%$ of the kinetic energy in original SWs were used to make the SSWs in the even and the odd cases, respectively. Interestingly, we found that the energy used to make SSWs in collisions against *hard* walls (i.e., where all forces acting on the wall is set to zero) was also $\approx 2.3\%$. Thus, in terms of SSW production, as one might expect, the hard-wall case behaves similarly as the odd-chain case. If the wall, however, is *softened*, production of more energetic SSWs, as observed experimentally by Job *et al.* [9], is expected.

To understand the dynamics of SW-SW collisions, we focus on the displacements and the kinetic energies of several grains in the vicinity of the collision point in the even and odd cases, respectively. Here we use $v_0=\pm 5\times 10^{-5}$. In Fig. 2(a) (even case) and Fig. 2(b) (odd case), we first consider grain displacements. In Fig. 2(a), we show the displacements of grains 248–253 in consecutive order with labels (i)–(vi). Time axis runs from right to left in Fig. 2(a). The SW-SW collision happens between grains 250 and 251. In the particular simulations shown here, the grains are barely in contact before the collision. Postcollision we find that grains 249 and 250, 250 and 251, and 251 and 252 lose contact. All the grains shown here move significantly after the collision process, indicative of the higher kinetic energy available imme-

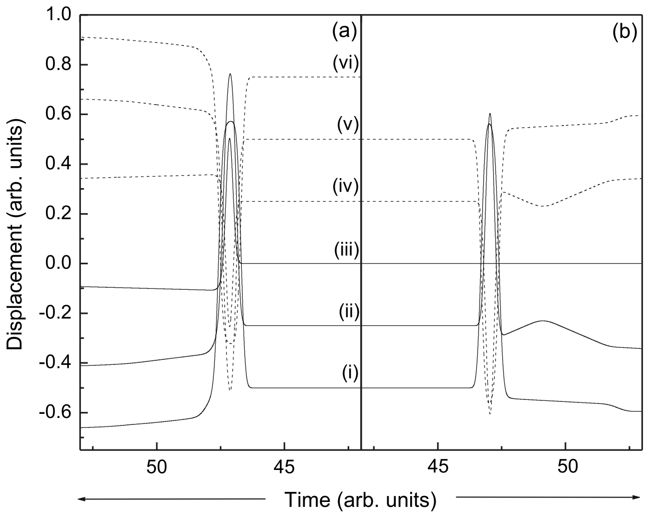

FIG. 2. Panels (a) and (b) show displacements of the grains in time in the collision region for even and odd systems, respectively. Data have been normalized and shifted in order to match the initial positions of grains in both systems (see text).

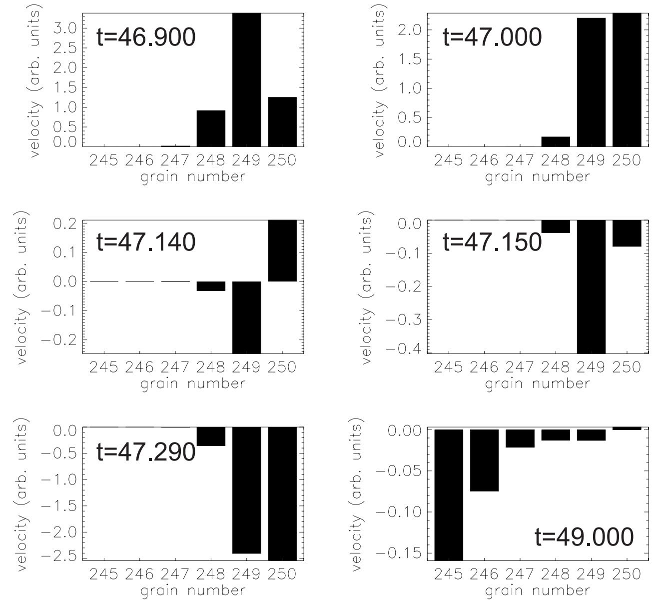

FIG. 3. The velocities of grains 245–250 are shown for the even chain at instants just before, during, and just after the collision. Grain 250 is the time-evolving edge grain here. Observe how the measurable velocities of four grains at $t=46.900$ gets *squeezed* (see text) and eventually ends up being shared by six grains at $t=49.000$. Compact support demands that SSWs form as such a wide energy bundle cannot be fitted within a single SW. The vertical axes have a multiplier of 10^{-5}.

diately after the collision process in edge collisions. The sharp and distinct time dependence of the slopes of the displacement-time graphs immediately after the collision is associated with the times at which SSWs form. The distinct rates of slope change immediately after the collision is needed to ensure that all the grains necessary to make a SSW can conspire to produce them.

In Fig. 2(b), we show the displacements of grains 248–252 in consecutive order with labels (i)–(v). Time axis runs from left to right in Fig. 2(b). In this odd chain, the central grain, grain 250, remains at rest at all times. In this case grains 249 and 250 and grains 250 and 251 lose contact postcollision. Grains 249 and 251 rattle between grains 250 and between 248 and 252, respectively, immediately following the collision. In both cases, the granular alignments do not return to their original equilibrium configurations.

It is not easy to visualize what happens physically to the energy bundle as it collides and immediately postcollision by just looking at Fig. 2. It is easiest perhaps to imagine that it is the same energy bundle that hits an edge and turns around. With that in mind, we have attempted to visualize the collision process in Figs. 3 and 4. In Fig. 3, we show the collision for the even chain and the time-sequenced velocity histograms clearly confirm our hypothesis that the SW gets squeezed in the course of the collision process. This means that the potential energy of what was the SW prior to the collision is now a squeezed energy bundle with higher potential energy and hence less kinetic energy than that warranted by the virial theorem. The bundle now must dilate as it turns around so as to relieve itself of the excess potential energy and convert it to kinetic energy. But a SW is the only way to transport energy through an unloaded granular chain and the SW can only be of a fixed width. Thus, the system has no choice but to generate SSWs. The results shown in Fig. 4 are very similar to those in Fig. 3 except that the times are chosen as appropriate for the odd-numbered system where the central grain, grain 250, remains static at all times.

The complex dynamics of the collision process is evident by looking at the velocity histogram at $t=47.042$ where the front of the SW is moving in opposition to the grains behind it. At $t=48.676$ we see that five grains are moving, and in the velocity scale in which the data are shown this means that SSW formation is required due to dilation of the original SW following the collision process.

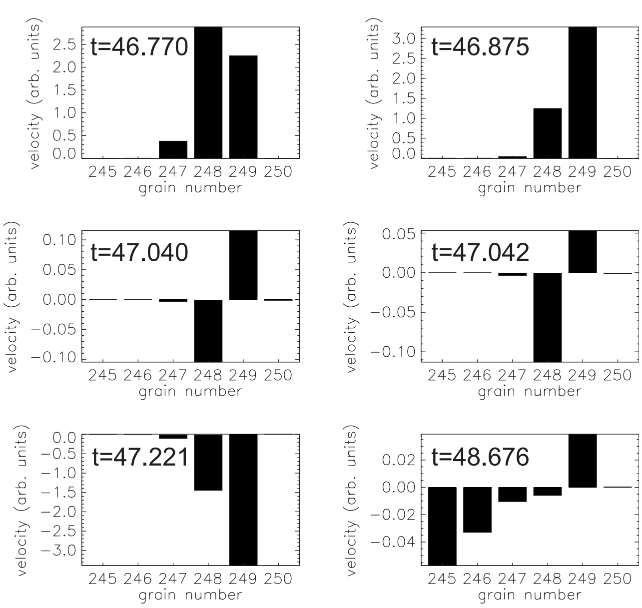

FIG. 4. The velocities of grains 245–250 are shown for the *odd* chain at instants just before, during, and just after the collision. The vertical axes have a multiplier of 10^{-5}.

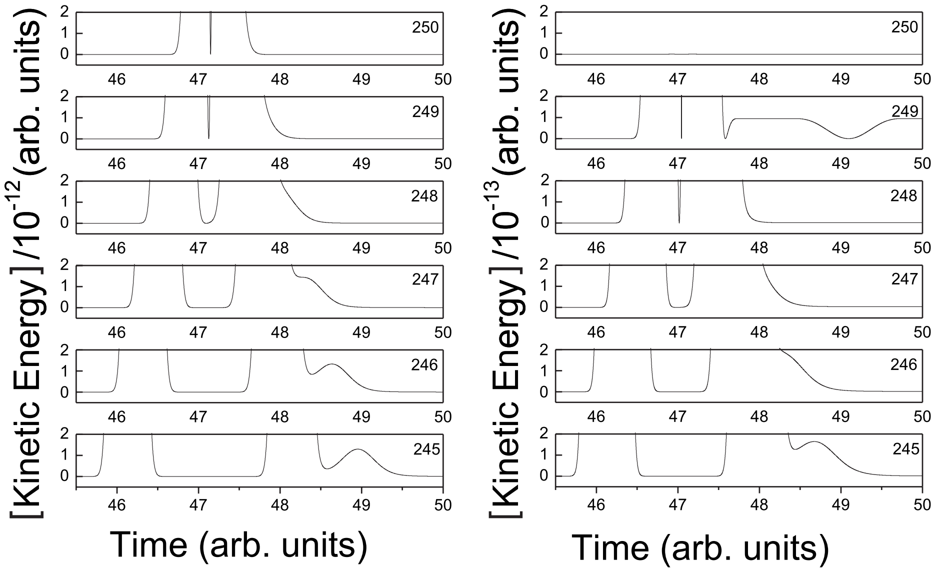

FIG. 5. The birth of the first SSW is shown for the even case in the left panel and for the odd case in the right panel (see text). Here $n=2.5$, $a=5657$, and $v_0=\pm 5\times 10^{-5}$.

The processes of SSW creation for edge and central grain collisions, respectively, are best captured in Fig. 5. The left panels describe edge collision in the even system and the right panels describe the odd-system collision case. Observe that grain 250 on the left panel begins to move as one SW reaches it, momentarily stops, and then moves as the second SW reaches it. As shown in Fig. 2, motion due to each of these SWs causes displacements in opposite directions. This feature can also be seen by observing the velocity histograms at $t=47.140$ and $t=47.200$ in Fig. 3. Further, the zeroing of kinetic energy shortly after $t=47$ for grain 250 implies that potential energy is stored in the grain (which is not shown explicitly here) during the collision time. Due to causality effects, grain 250 is unable to return to its original equilibrium position as soon as the SWs cross as shown in Fig. 2. Observe in Fig. 5 that the total time across which grain 250 moves here exceeds the typical time of motion of a grain due to the passage of a SW. We contend that the grain gets simultaneously squeezed from both directions during collision and hence ends up possessing the potential energy across a longer time than that needed during the passage of a single solitary wave through a grain. Because a nonlinear system under zero-loading condition such as this can only propagate energy through the grains via SWs, grain 250 must now discharge its energy by forming one or more SWs (in each direction). However, generation of a single SW by a grain that holds potential energy because of compression from both sides requires additional time, and hence more space (since t and z are related by c). However, each SW has fixed width. Thus, grain 250 must *first* transfer its energy to initiate the formation of a single SW, which is the largest SW it can generate with the kinetic energy it possesses. Indeed this is what is seen in all the existing simulations (see the review by Sen *et al.* in [8]). This SW can only carry the allowed amount of potential energy. SSWs, one or typically more, must now form to discharge the remaining energy in the same manner. Qualitatively speaking, this is why we see that grain 245 (shown in the bottom panels of Fig. 5) is about to give birth to a SSW. The distance scale associated with the formation of the first SSW (see the dynamics of grain 245) approximately relates to the distance scale associated with the conversion of an arbitrary amount of energy into a SW, which is estimated to be ~7–10 grains [11].

The dynamics of grain 250 in the odd chain reflects no motion as the two SWs pass through it. However, grain 249 (and 251; not shown) behaves somewhat similarly to grain 250 in the even chain. In both cases, a SSW is born past grain 245. The first SSW is born earlier in time in the even (left panel) chain. A distinguishing feature in the dynamics of grain 249 is the "flatline" behavior immediately after the collision. To understand its origin observe that immediately after an edge collision the grains end up moving in opposite directions and lose contact. This is when there is no net force on them and hence the velocity and the kinetic energy are constants [see Fig. 2(b)]. Because no potential energy can be stored at the grain edge, the crossing of SWs can be compared to a SW hitting an infinitely hard wall. The only possibility now is for the leading grain, which just hit the wall, to rebound without any loss of kinetic energy, while the others in the original SW still moves forward. The energy involved in this "train wreck" must be harnessed into SWs in order to construct a viable solution to the equations of motion (which do not admit stable entities other than SWs). Thus, the system immediately starts to remake a large SW and SSWs, as needed. The difference between this case and the odd-chain case is that the grains now have more kinetic energy than in the previous case. This system hence makes SWs and SSWs that carry more kinetic and potential energies and move hence move faster (as seen in Table I).

VI. GRANULAR CHAINS WITH SOFTER INTERACTION

In the above discussions we have focused on granular alignments with $n=2.5$. It is not unusual [12] to think in

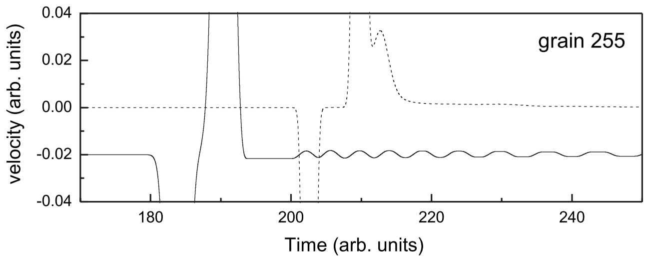

FIG. 6. Velocity of grain 255—as function of time—using two different intergrain interactions, $n=2.5$ and $n=2.1$, and an even chain. For clarity, data for $n=2.1$ (solid line) have been shifted by -0.02 in the y axis. Dashed line shows data with $n=2.5$.

terms of alignments where grain-grain interactions are Hertz type except that an arbitrary n (>2) is employed. When $n \to \infty$, the SWs are 1 grain diameter wide [18] and progressively smaller amounts of potential energy will be involved in the collisions. Hence SSW production will be suppressed. The interesting regime hence is when $n \to 2$, when the potential energy of the propagating pulse approaches the value of the kinetic energy and hence approaches half the total energy.

We set $a=1$ and $v_0=\pm 1$ in our numerical calculations, then for $\delta_{i,i+1}<1$, the $n=2.1$ system stores *more* potential energy than the $n=2.5$ system. This means that a colliding SW can be squeezed more during collisions. Since the potential energy released from squeezing will also get converted to kinetic energy over extended times compared to the $n=2.5$ case here, one may expect long-lived dynamics associated with the SW collision process in the collision region. In Fig. 6 we show the velocity vs time behavior of a chosen grain near the SW-SW collision point. We show that while the single SSW created immediately after the collision in the $n=2.5$ system carries more energy, the softer $n=2.1$ system generates a large number of small SSWs, somewhat like that of acoustic oscillations that would only be possible for a $n=2$ system. The energy fraction used to make SSWs in the $n=2.5$ system is 0.2%, while this proportion is only 0.01% in the $n=2.1$ system. Thus, in addition to manipulating edge versus central grain collisions, SSW production may be manipulated by considering systems with different n values.

VII. SUMMARY AND CONCLUSIONS

Here we have revisited the problem of collision of identical and opposite propagating SWs in unloaded granular alignments where the collision happens at the contact point between two adjacent grains. In that sense, this work is an extension of the work in Ref. [10], where such collisions were probed when the SWs met at the center of a grain. The purpose of this work, however, is broader. Here we establish that depending upon the exact position of the collision point of two identical and opposite propagating SWs in a granular alignment, the resultant SSWs can carry significantly (in this case as much as nine times) more energy.

SSWs typically carry very small amounts of energy and are hence difficult to detect in laboratory experiments unless the elastic properties of the wall against which a SW is colliding are manipulated [9]. Here we show that a significant enhancement in SSW energies can be brought about *without* introducing walls with different elastic constants. Further, this work also sheds light on *why* the SSWs must form. We have presented simple physics-based arguments involving the virial theorem and squeezing and subsequent dilation of a soft energy bundle to construct a qualitative rationale for their formation. The simulation-based analyses presented here confirm the arguments and offer insights into ways to further enhance the energy carried by SSWs by rendering the SWs more squeezable and insuring that the maximum possible number of grains can participate in the spatial dilation of the returning energy bundle following the collision.

Anytime SWs collide, SWs will be squeezed and hence SSW formation must follow. However, the energy content of the SSWs can vary significantly depending upon whether the SWs are identical or otherwise, moving in opposition or in parallel, and, of course, the point at which the collision occurs. In an idealized granular alignment with zero dissipation and held within boundaries, a single SW initiated by a perturbation at $t=0$ will continue to break and produce SSWs. Recent work [19] strongly suggests that in certain collisions SWs can also grow in energy after collision. Our earlier work has already suggested that SW breakdown and SW growth processes eventually strike a balance in the time evolution of these systems (see the review paper of Sen *et al.* in Ref. [8]). At large enough times, the system tends to reach an equilibriumlike (quasiequilibrium) phase with sustained large energy fluctuations and work on such a phase has been reported elsewhere [20,21]. The present work lends different insights into the nature of energy-exchange processes during the course of many-body interactions in purely nonlinear systems and may have implications on tuning the relaxation time to quasiequilibrium in purely nonlinear systems.

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